Bump Mapping from a Normal Map to a Scene Pixel

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Given the normal **n** (normalized) of a scene pixel and a normal map, where the *(u, v)* coordinates of the pixel on the map are known, we should find the rotation matrix **R** around an arbitrary axis **e** such that the rotated z axis of the normal map coincides with **n**.

So, the axis of rotation must be the cross product between the z axis and the normal **n**.

$$e = z\,x\,n = (0,0,1)^T\ x\,(n_x, n_y, n_x)^T = (-n_y, n_x, 0)^T$$

Normalizing,

$$\boxed{\bar{e} = (\bar{e}_x, \bar{e}_y, \bar{e}_z)^T = \left(\frac{-n_y}{\sqrt{n_y^2 + n_x^2}}, \frac{n_x}{\sqrt{n_y^2 + n_x^2}}, 0 \right)^T}$$

Now, we should find the rotation matrix such that,

$$R_{\bar{e}}(\phi).z = n$$

$$R_{\bar{e}}(\phi).\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix}$$

Rotation matrix around an arbitrary **e** axis is defined as:

$$R_{\bar{e}}(\phi) = \begin{pmatrix} \cos\phi + (1-\cos\phi)\bar{e}_x^2 & (1-\cos\phi)\bar{e}_x\bar{e}_y - \bar{e}_z\sin\phi & (1-\cos\phi)\bar{e}_x\bar{e}_z + \bar{e}_y\sin\phi \\ (1-\cos\phi)\bar{e}_x\bar{e}_y + \bar{e}_z\sin\phi & \cos\phi + (1-\cos\phi)\bar{e}_y^2 & (1-\cos\phi)\bar{e}_y\bar{e}_z - \bar{e}_x\sin\phi \\ (1-\cos\phi)\bar{e}_x\bar{e}_z - \bar{e}_y\sin\phi & (1-\cos\phi)\bar{e}_y\bar{e}_z + \bar{e}_x\sin\phi & \cos\phi + (1-\cos\phi)\bar{e}_z^2 \end{pmatrix}$$

So,

$$R_{\bar{e}}(\phi).\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} n_x \\ n_y \\ n_z \end{pmatrix} \Leftrightarrow R_{\bar{e}}(\phi) = \begin{pmatrix} r_{00} & r_{01} & n_x \\ r_{10} & r_{11} & n_y \\ r_{20} & r_{21} & n_z \end{pmatrix}$$

Then,

$$\cos\phi + (1-\cos\phi)\bar{e}_z^2 = n_z$$

$$\cos\phi + \bar{e}_z^2 - \bar{e}_z^2\cos\phi = n_z$$

$$(1-\bar{e}_z^2)\cos\phi = n_z - \bar{e}_z^2$$

$$\cos\phi = \frac{n_z - \bar{e}_z^{\,2}}{1 - \bar{e}_z^{\,2}}$$

Replacing what we know from the normalized axis of rotation,

$$\boxed{\cos\phi = n_z}$$

Now we want to obtain the sinus,

$$(1-\cos\phi)\bar{e}_y\bar{e}_z - \bar{e}_x \sin\phi = n_y$$
$$(1-n_z)\bar{e}_y\bar{e}_z - \bar{e}_x \sin\phi = n_y$$
$$(1-n_z)\bar{e}_y\bar{e}_z - n_y = \bar{e}_x \sin\phi$$
$$\frac{-n_y}{\bar{e}_x} = \sin\phi$$

Analogous for n_x,

$$(1-\cos\phi)\bar{e}_x\bar{e}_z + \bar{e}_y \sin\phi = n_x$$
$$(1-n_z)\bar{e}_x\bar{e}_z + \bar{e}_y \sin\phi = n_x$$
$$n_x - (1-n_z)\bar{e}_x\bar{e}_z = \bar{e}_y \sin\phi$$
$$\frac{n_x}{\bar{e}_y} = \sin\phi$$

So,

$$\boxed{\sin\phi = \frac{-n_y}{\bar{e}_x} = \frac{n_x}{\bar{e}_y} = \sqrt{n_y^{\,2} + n_y^{\,2}}}$$

We can now write the final rotation matrix,

$$\boxed{R_{\bar{e}}(\phi) = \begin{pmatrix} n_z + (1-n_z)\bar{e}_x^{\,2} & (1-n_z)\bar{e}_x\bar{e}_y & n_x \\ (1-n_z)\bar{e}_x\bar{e}_y & n_z + (1-n_z)\bar{e}_y^{\,2} & n_y \\ -n_x & -n_y & n_z \end{pmatrix}}$$

Given the normal **b** at coordinates (u,v) of the normal map, the new pixel normal **n’** is calculated as follows,

$$\boxed{R_{\bar{e}}(\phi).\begin{pmatrix} b_x \\ b_y \\ b_z \end{pmatrix} = \begin{pmatrix} n'_x \\ n'_y \\ n'_z \end{pmatrix}}$$